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LIE IDEALS AND GENERALIZED JORDAN SEMIDERIVATIONS

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Abstract

Let R be an associative ring. An additive mapping f : R ! R is called a generalized semiderivation if there exists a semiderivation d associated with map g such that (i) f (xy) = f (x)y + g(x)d(y) = f (x)g(y) + xd(y); (ii) f (g(x)) = g(f (x)); for all x; y 2 R: In this paper, we prove that if R is a prime ring with characteristic deferent from two, g an endomorphism of R; f a generalized Jordan semiderivation associated with d and g on a Lie ideal of R; then f is a generalized semiderivation and d is a semiderivation on a nonzero Lie ideal of R:

1. INTRODUCTİON

Throughout R will present an associative prime ring with center Z and characterisitc different from two. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$. Recall that a ring R is prime if $xRy = (0)$ implies $x = 0$ or $y = 0$. An additive subgroup U of R is said to be a Lie ideal of R, if $[u, r] \in U$, for all $u \in U$, $r \in R$. An additive mapping $d: R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. Also, d is called Jordan derivation if $d(x^2) = d(x)x + xd(x)$, for all $x \in R$. Clearly every Jordan derivation is a special case of a derivation with $y = x$. The converse is, in general not true. A well known result due to Herstein [6] shows that every Jordan derivation on a 2-torsion free prime ring is a derivation. This result was generalized in [3] and [4]. Latter on, the authors proved this result for Lie ideals of R such that $u^2 \in U$ for all $u \in U$ in [1] and [7].

In [2], Bergen has introduced the following notion: An additive mapping $d: R \to R$ is called a semiderivation if there exists a function $g: R \to R$ such that (i) $d(xy) = d(x)g(y) + xd(y) = d(x)y + g(x)d(y)$ and (ii) $d(g(x)) = g(d(x))$ hold for all $x, y \in R$. In case g is an identity map of R, then all semiderivations associated with g are merely derivations. On the other hand, if g other main motivating examples are of the form $d = g - 1$ where g is any ring endomorphism of R such that $g \neq 1$. Then d is a semiderivation with associated map g which is not a derivation.

In [5], the authors introduced generalized semiderivation as follows: An additive mapping $f: R \to R$ is called a generalized semiderivation if there exists a semiderivation d associated with g such that (i) $f(xy) = f(x)y + g(x)d(y) = f(x)g(y) + xd(y)$ and (ii) $f(g(x)) = g(f(x))$ for all $x, y \in R$. Motivated by the concept of Jordan derivations and generalized Jordan derivations, the concept of Jordan semiderivations and generalized Jordan semiderivations were investigated for rings in this paper.

In our study, following the line of investigation of previous cited results, we prove that if R is a prime ring with characteristic different from two and U a Lie ideal of R such that $u^2 \in U$ for all $u \in U$ and f a generalized Jordan semiderivation associated with d and g on U, then f is a generalized semiderivation and d is a semiderivation on U.

2. RESULTS

Lemma 1 [6, Lemma 1]Let R be a semiprime 2-torsion free ring and

U a nonzero Lie ideal of R. Suppose that $[U,U]\subset Z$, then $U\subseteq Z$.

Lemma 2 [2, Lemma 4]Let R be a prime ring with characteristic not two, $a,b\in R$. If U a noncentral Lie ideal of R and $aUb=(0)$, then $a=0$ or $b=0$.

Lemma 3 Let R be a prime ring with characteristic different from two and U a Lie ideal of R such that $u^2\in U$ for all $u\in U$. If R admits a nonzero generalized Jordan semiderivation (f,d) associated with d and g, then

i) $f(uv+vu)=f(u)v+g(u)d(v)+f(v)u+g(v)d(u)$, for all $u,v\in U$.

ii) $f(uvu)=f(u)vu+g(u)d(v)u+g(u)g(v)d(u)$, for all $u,v\in U$.

iii) $f(uvw+wvu)=f(u)vw+g(u)d(v)w+g(u)g(v)d(w)+f(w)vu+g(w)d(v)u+g(w)g(v)d(u)$, for all $u,v,w\in U$.

Proof. Results are easily obtained in the way to that in [5, Lemma 2.3].

Corollary 1 Let R be a prime ring with characteristic different from two and U a Lie ideal of R such that $u^2\in U$ for all $u\in U$. If R admits a nonzero Jordan semiderivation d associated with g, then

i) $d(uv+vu)=d(u)v+g(u)d(v)+d(v)u+g(v)d(u)$, for all $u,v\in U$.

ii) $d(uvu)=d(u)vu+g(u)d(v)u+g(u)g(v)d(u)$, for all $u,v\in U$.

iii) $d(uvw+wvu)=d(u)vw+g(u)d(v)w+g(u)g(v)d(w)+d(w)vu+g(w)d(v)u+g(w)g(v)d(u)$, for all $u,v,w\in U$.

Proof. If we take $f=d$ in Lemma 3, we get the required result.

Proceeding on the same lines with necessary variations in proof of [5, Lemma 2.3] and using the definition of generalized Jordan semiderivation, we can prove the following Lemma.

Lemma 4 Let R be a prime ring with characteristic different from two and U a Lie ideal of R such that $u^2\in U$ for all $u\in U$. If R admits a nonzero generalized Jordan semiderivation (f,d) associated with d and g, then

i) $f(uv+vu)=f(u)g(v)+ud(v)+f(v)g(u)+vd(u)$, for all $u,v\in U$.

ii) $f(uvu)=f(u)g(v)g(u)+ud(v)g(u)+uvd(u)$, for all $u,v\in U$.

iii) $f(uvw+wvu)=f(u)g(v)g(w)+ud(v)g(w)+f(w)g(v)g(u)+wd(v)g(u)$, for all $u,v,w\in U$.

Proof. i) By the definition of f, we get for all $u,v\in U$,

$$f((u+v)^2)=f((u+v)(u+v))=f(u^2)+f(uv+vu)+f(v^2) \quad (2.1)$$

$$=f(u)g(u)+ud(u)+f(uv+vu)+f(v)g(v)+vd(v).$$

On the other hand,

$$f((u+v)^2)=f(u+v)g(u+v)+(u+v)d(u+v) \quad (2.2)$$

$$=f(u)g(u)+f(u)g(v)+f(v)g(u)+f(v)g(v)+ud(u)+ud(v)+vd(u)+vd(v).$$

Comparing (2.1) and (2.2), we have

$$f(uv+vu)=f(u)g(v)+ud(v)+f(v)g(u)+vd(u)$$

for all $u,v\in U$.

ii) Replacing v by $uv+vu$ in (i), we get

$$f(u(uv+vu)+(uv+vu)u)=f(u)g(uv+vu)+ud(uv+vu)+f(uv+vu)g(u)+(uv+vu)d(u).$$

We can use (i) taking $f=d$ in the last equation. Hence we get

$$f(u(uv+vu)+(uv+vu)u)=f(u)g(uv)+f(u)g(vu)+ud(u)g(v) \quad (2.3)$$

$$+u^2d(v)+ud(v)g(u)+uvd(u)+f(u)g(v)g(u)+ud(v)g(u)+f(v)g(u)g(u)+vd(u)g(u)+uvd(u)+vud(u).$$

On the other hand,

$$f(u(uv+vu)+(uv+vu)u)=f(u^2v+uvu+uvu+vu^2)=f(u^2v+vu^2)+2f(uvu).$$

Again using (i) in the last equation, we have

$$f(u(uv+vu)+(uv+vu)u)=f(u^2)g(v)+u^2d(v)+f(v)g(u^2)+vd(u^2)+2f(uvu)$$

$$=f(u)g(u)g(v)+ud(u)g(v)+u^2d(v)+f(v)g(u^2)+vd(u)g(u)+vud(u)+2f(uvu).$$

Comparing (2.3) and (2.4) and using $charR\neq 2$, we find that

$$f(uvu)=f(u)g(v)g(u)+ud(v)g(u)+uvd(u),$$

for all $u,v\in U$.

iii) Linearizing (ii) on u, we get

$$f\big((u+w)v(u+w)\big)=f(u+w)g(v)g(u+w)+(u+w)d(v)g(u+w)+(u+w)vd(u+w) \quad (2.5)$$

$$=f(u)g(v)g(u)+f(u)g(v)g(w)+f(w)g(v)g(u)+f(w)g(v)g(w)$$

$$+ud(v)g(u)+ud(v)g(w)+wd(v)g(u)+wd(v)g(w)+uvd(u)+uvd(w)+wvd(u)+wvd(w).$$

On the other hand,

$$f((u+w)v(u+w))=f(uvu)+f(uvw+wvu)+f(wvw)$$

and using (ii),

$$f((u+w)v(u+w))=f(u)g(v)g(u)+ud(v)g(u)+uvd(u)+f(uvw+wvu)+f(w)g(v)g(w)+wd(v)g(w)+wvd(w). \quad (2.6)$$

Comparing (2.5) and (2.6) and using $charR\neq 2$, we find that

$$f(uvw+wvu)=f(u)g(v)g(w)+ud(v)g(w)+uvd(w)+f(w)g(v)g(u)+wd(v)g(u)+wvd(u),$$

for all $u,v,w\in U$. (2.1)

Corollary 2 Let R be a prime ring with characteristic different from two and U a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. If R admits a nonzero Jordan semiderivation d associated with g, then

i) $d(uv+vu) = d(u)g(v) + ud(v) + d(v)g(u) + vd(u)$, for all $u, v \in U$.

ii) $d(uvu) = d(u)g(v)g(u) + ud(v)g(u) + uvd(u)$, for all $u, v \in U$.

iii) $d(uvw+wvu) = d(u)g(v)g(w) + ud(v)g(w) + d(w)g(v)g(u) + wd(v)g(u)$, for all $u, v, w \in U$.

Remark 1 We introduce abbreviations

$$\delta(u,v) = f(uv) - f(u)v - g(u)d(v),$$
$$\phi_u(v) = d(uv) - d(u)v - g(u)d(v),$$
$$\rho(u,v) = f(uv) - f(u)g(v) - ud(v)$$
$$\psi_u(v) = d(uv) - d(u)g(v) - ud(v),$$

for all $u, v, w \in U$.

Observe also by the definition of generalized semiderivation we have

$$\delta(u,v) = 0 \Leftrightarrow \rho(u,v) = 0.$$

On the other hand, by Lemma 3, Lemma 4 (i) and the corollaries of them, we get

$$\delta(u,v) = -\delta(v,u),$$
$$\rho(u,v) = -\rho(v,u),$$
$$\phi_u(v) = -\phi_v(u),$$
$$\psi_u(v) = -\psi_v(u)$$

and

$$\delta(u, v+w) = \delta(u,v) + \delta(u,w),$$
$$\rho(u, v+w) = \rho(u,v) + \rho(u,w),$$
$$\phi_u(v+w) = \phi_u(v) + \phi_u(w),$$
$$\psi_u(v+w) = \psi_u(v) + \psi_u(w),$$

for all $u, v, w \in U$.

Lemma 5 Let R be a prime ring with characteristic different from two and U a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. If R admits a nonzero generalized Jordan semiderivation (f,d) associated with d and g, then

$$\delta(u,v)w[u,v] + g([u,v])g(w)\phi_u(v) = 0,$$

for all $u, v, w \in U$.

Proof. Consider $W = f(uvwvu + vuwuv)$, for all $u, v, w \in U$. Using Lemma 3 (iii), we get

$$W = f(uvwvu + vuwuv) = f((uv)w(vu) + (vu)w(uv)) \qquad (2.7)$$
$$= f(uv)wvu + g(uv)d(w)vu + g(uv)g(w)d(vu)$$
$$+ f(vu)wuv + g(vu)d(w)uv + g(vu)g(w)d(uv).$$

On the other hand, according to Lemma 3 (ii) and its corollary, we see that

$$W = f(uvwvu + vuwuv) = f(u(vwv)u) + f(v(uwu)v)$$
$$= f(u)vwvu + g(u)d(vwv)u + g(u)g(vwv)d(u)$$
$$+ f(v)uwuv + g(v)d(uwu)v + g(v)g(uwu)d(v)$$

and so

$$W = f(u)vwvu + g(u)d(v)wvu + g(u)g(v)d(w)vu \qquad (2.8)$$
$$+ g(u)g(v)g(w)d(v)u + g(u)g(vwv)d(u)$$
$$+ f(v)uwuv + g(v)d(u)wuv + g(v)g(u)d(w)uv$$
$$+ g(v)g(u)g(w)d(u)v + g(v)g(uwu)d(v).$$

By comparing these two equations, we obtain that

$\delta(u,v)w[u,v] + g([u,v])g(w)\phi_u(v) = 0$, for all $u, v, w \in U$.

Lemma 6 Let R be a prime ring with characteristic different from two and U a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. If R admits a nonzero generalized Jordan semiderivation (f,d) associated with d and g, then

$$\rho(u,v)g(w)g([u,v]) = 0, \text{for all } u, v, w \in U.$$

Proof. Consider $Y = f(uvwvu + vuwuv)$, for all $u, v, w \in U$. Using Lemma 4 (iii), we get

$$Y = f(uvwvu + vuwuv) = f((uv)w(vu) + (vu)w(uv)) \qquad (2.9)$$
$$= f(uv)g(w)g(vu) + uvd(w)g(vu) + uvwd(vu)$$
$$+ f(vu)g(w)g(uv) + vud(w)g(uv) + vuwd(uv).$$

On the other hand, according to Lemma 4 (ii) and its corollary, we have

$$Y = f(uvwvu + vuwuv) = f(u(vwv)u) + f(v(uwu)v)$$
$$= f(u)g(vwv)g(u) + ud(vwv)g(u) + uvwvd(u)$$
$$+ f(v)g(uwu)g(v) + vd(uwu)g(v)v + vuwud(v)$$

and so

$$Y = f(u)g(vwv)g(u) + ud(v)g(w)g(v)g(u) \qquad (2.10)$$
$$+ uvd(w)vg(u) + uvwd(v)g(u) + uvwvd(u)$$
$$+ f(v)g(uwu)g(v) + vd(u)g(w)g(u)g(v)$$
$$+ vud(w)g(u)g(v) + vuwd(u)g(v) + vuwud(v).$$

Comparing (2.9) and (2.10), we arrive at

$$\rho(u,v)g(w)g([u,v]) = 0, \text{for all } u, v, w \in U.$$

The following theorem gives a generalization of Herstein's well known result [6, Theorem 4.1] and a extension of [5, Theorem].

Theorem 1 Let R be a prime ring with characteristic different from two and U a noncentral Lie ideal of R such that $u^2 \in U$ for all $u \in U$. If R admits a nonzero generalized Jordan semiderivation (f,d) associated with d and g, then R admits a nonzero generalized semiderivation (f,d) associated with d and g.

Proof. Our aim is to show $\delta(u,v) = 0$ for all $u, v \in U$. By Lemma 6, we get

$$\rho(u,v)g(w)g([u,v]) = 0, \text{for all } u, v, w \in U.$$

It is clear that $T = g(U)$ is a Lie ideal of R. Lemma 2 yields that for each $u \in U$ either $\rho(u,v) = 0$ or $g([u,v]) = 0$, for all $v \in U$. For each $u \in U$, let

$K = \{v \in U | \rho(u,v) = 0\}$

and

$L = \{v \in U | g([u,v]) = 0\}$.

K and L are additive subgroups of U whose union is U. By Brauer's trick, we get either $U = K$ or $U = L$.

Assume that $U = K$. Hence we arrive at $\rho(u,v) = 0$, and so $\delta(u,v) = 0$. Hence our proof is completed.

Now we get $U = L$. Then we have $g([u,v]) = 0$, for all $u, v \in U$. By Lemma 5, we know that $\delta(u,v)w[u,v] + g([u,v])g(w)\phi_u(v) = 0$. Using $g([u,v]) = 0$ in this equation, we have $\delta(u,v)w[u,v] = 0$, for all $w \in U$. Applying Lemma 2 and using the same method again, we find that $[u,v] = 0$, for all $v \in U$ or $\delta(u,v) = 0$. If $[u,v] = 0$, for all $v \in U$, then we get $U \subseteq Z$, by Lemma 1, a contradiction. So, we must obtain that $\delta(u,v) = 0$, and so, R admits a generalized semiderivation (f,d) associated with d and g. The proof is completed.

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